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Bilabelled increasing trees and hook-length formulae

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ABSTRACT

We introduce two different kinds of increasing bilabellings of trees, for which we provide enumeration formulae. One of the bilabelled tree families considered is enumerated by the reduced tangent numbers and is in bijection with a tree family introduced by Poupard [11]. Both increasing bilabellings naturally lead to hook-length formulae for trees and forests; in particular, one construction gives a combinatorial interpretation of a formula for labelled unordered forests obtained recently by Chen et al. [1].

1. Introduction

This paper has two main concerns: first we want to give combinatorial interpretations and proofs of certain hook-length formulae for trees and forests, respectively; we do this by introducing two kinds of increasing bilabellings of trees, which seem to be of interest on its own, and thus yielding a second aspect.

Given a rooted tree T (since throughout this paper we will always consider rooted trees, from now on we will not express this explicitly), we will call a node $u \in T$ a *descendant* of node $v \in T$ if v is lying on the unique path from the root of T to u. The *hook-length* $h_v := h(v)$ of a node $v \in T$ is defined as the number of descendants of v including the node v itself (i.e., it is the size of the subtree rooted at v).

Let us denote by $\mathcal{U}$ the family of *labelled unordered trees*, i.e., the set of trees T, whose vertices are labelled by distinct integers of $\{1, 2, \dots, |T|\}$, with $|T|$ the size of T (measured by the number of vertices). When using the term unordered tree we assume that there is no left-to-right ordering on the children of any node, i.e., we consider such a tree as a root-vertex, where a set of subtrees is attached. Conversely, for an ordered tree we assume that there is a linear ordering on the children of each node.

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Furthermore, with $\mathcal{U}(n)$ we denote the set of trees $T \in \mathcal{U}$ of size $|T| = n$; more generally, for a family $\mathcal{C}$ of combinatorial objects, with $\mathcal{C}(n)$ we always denote the set of objects of $\mathcal{C}$ of size n.

Having introduced this notation we can state the following hook-length formulae for labelled unordered trees, which will be shown afterwards.

Theorem 1. *The following hook-length formulae for the family $\mathcal{U}$ of labelled unordered trees hold (with $n \geq 1$):*

$$\sum_{T\in\mathcal{U}(n)} \prod_{v\in T} \left(\frac{1}{2h_v(2h_v-1)}\right) = \frac{n!}{(2n)!}\tilde{E}_n, \tag{1}$$

$$\sum_{T\in\mathcal{U}(n)} \prod_{v\in T} \frac{1}{h_v^2} = \frac{(n-1)!}{2^{n-1}}. \tag{2}$$

Here $\tilde{E}_n$ denotes the so-called *reduced tangent numbers*, which might be defined via their generating function: $\sum_{n\geq 1} \tilde{E}_n \frac{z^{2n-1}}{(2n-1)!} = \sqrt{2}\tan\left(\frac{z}{\sqrt{2}}\right)$; it is well known that these numbers appear in the enumeration of various combinatorial objects, see, e.g., [6,11].

Recently various hook-length formulae for different tree families and forests of trees could be obtained, see, e.g., [1,7,8,10]. In particular, a so-called "expansion technique" developed by Han [8] and generalized further by Chen et al. [1] and by the authors [9], which allows to determine the "hook-weight function" $\rho(n)$ from the generating function (or, when considering labelled tree families, the corresponding exponential generating function) $G(z) = \sum_{n\geq 1}\left(\sum_{T\in\mathcal{T}(n)}\prod_{v\in T}\rho(h_v)\right)z^n$, with $\mathcal{T}(n)$ the set of trees of size n of a family $\mathcal{T}$, turns out to be very fruitful. e.g., by using this expansion technique, the hook-length formula $\sum_{T\in\mathcal{B}(n)}\prod_{v\in T}\left(1+\frac{1}{h_v}\right) = \frac{2^n(n+1)^{n-1}}{n!}$ for the family of *binary trees* $\mathcal{B}$ obtained by Postnikov [10], which can be considered as the starting point of this research direction, could be shown rather easily.

However, besides the search and derivation of such hook-length formulae for trees a second important research aspect is to give combinatorial interpretations (or probabilistic interpretations, see [12]) of them and thus to obtain a "concrete meaning". e.g., the papers [2,13] present combinatorial proofs of the before-mentioned formula of Postnikov for binary trees. Another hook-length formula for the family of labelled unordered forests closely related to formula (2) appears in [1], where it has been obtained via the expansion technique, but the authors ask for a combinatorial interpretation of it; to provide such an interpretation can be considered as the origin of this paper.

Let us denote by $\mathcal{U}_F$ the family of *labelled unordered forests*, i.e., the set of forests $F = \{T_1, \dots, T_s\}$, with $s \geq 0$, where $T_1, \dots, T_s$ are labelled unordered trees, whose vertices are labelled by distinct integers of $\{1, 2, \dots, |F|\}$, with $|F| := |T_1| + \cdots + |T_s|$ the size of F. As we shall figure out later in Section 4 each hook-length formula for the family $\mathcal{U}$ of labelled unordered trees corresponds naturally to a hook-length formula for the family $\mathcal{U}_F$ of labelled unordered forests and vice versa. In particular, in Section 4 we state (as Eqs. (14) and (15), respectively) the hook-length formulae for the family of forests $\mathcal{U}_F$ corresponding to (1)–(2). As mentioned before Eq. (15) (which is equivalent to formula (2)) appears in [1], whereas (1) (and the equivalent formula (14)) seem to be new to the best of our knowledge. Since formulae for forests $\mathcal{U}_F$ immediately follow from the corresponding ones for trees $\mathcal{U}$ our main focus in this work will be given on trees, whereas results for forests and relations to trees will only be collected in Section 4. We want to remark that meanwhile there has been given in [4] another combinatorial interpretation of Eq. (15) by introducing hook-lengths on permutations.

Increasingly labelled trees, i.e., labelled trees, where the label of a child node is always larger than the label of its parent node, appear frequently in combinatorial contexts (e.g., in connection with hook-length formulae they occur in [7,12]), but they are also of interest in problems stemming from probability theory or computer science; see, e.g., [5] and references therein. In the present paper we introduce two different kinds of "bilabelled increasing trees"; in such trees each node always gets two

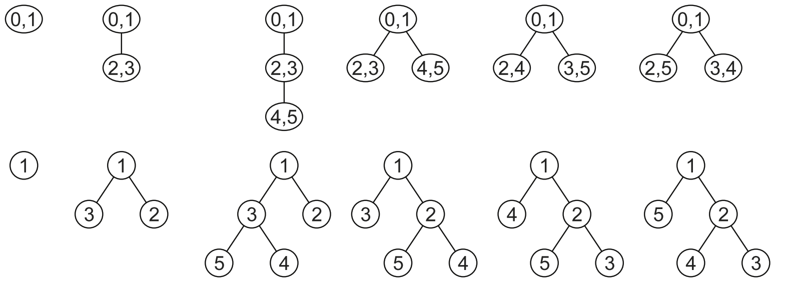


Fig. 1. Increasingly bilabelled unordered trees of size up to three and increasingly labelled strict binary trees of size up to five satisfying the "right-smallest property". The trees of the second row are obtained by applying the bijection φ as described in the bijective proof of Theorem 2 to the corresponding trees of the first row.

labels and the labelling will be of such a kind that the labelling of a child node is always larger (in a sense specified later) than the labelling of its parent node.

In order to prove formula (1) we consider in Section 2 trees, where each node gets a set of two different labels, the label sets of different nodes are disjoint, and where each label of a child node is always larger than any label of its parent node. We call such trees "increasingly bilabelled trees". To show formula (2) we consider in Section 3 trees, where each node gets an ordered pair of labels (we might think about a left and a right label), such that all left labels are different, all right labels are different, and where the left label of a child node is always larger than the left label of its parent node as well as the right label of a child node is always larger than the right label of its parent node. We call such trees "double increasingly labelled trees".

In the present paper we will be interested exclusively in specifically bilabelled *unordered* trees, but we remark that variants of such tree families could be used to give combinatorial interpretations of certain statistics defined on two random objects. e.g., the problem of computing (asymptotically) the probability that two binary search trees (generated by two random permutations of the same length) give the same binary tree (a problem which was open for some time and has been solved only recently in [3]) could be formulated also in terms of enumeration of double increasingly labelled binary trees.

2. Increasingly bilabelled trees

2.1. Enumeration results

We call a tree T a *bilabelled* tree, if each node $v \in T$ has got a set $\ell_B(v) = \{\ell^{[1]}(v), \ell^{[2]}(v)\}$ of two different integers (we may always assume that $\ell^{[1]}(v) < \ell^{[2]}(v)$), and where furthermore the label sets of different nodes are disjoint, i.e., $\ell_B(v) \cap \ell_B(w) = \emptyset$, for $v \neq w$. We say then that T is a bilabelled tree with *label set* $\mathcal{M} = \mathcal{M}(T) = \bigcup_{v \in T} \ell_B(v)$; of course, $|\mathcal{M}| = 2n$, for a tree T of size $|T| = n$. A bilabelled tree T is called *increasing*, if it holds that each label of a child node is always larger than all labels of its parent node: $\ell_B(v) \prec \ell_B(w)$, whenever w is a child of v, where we use the relation $\{a^{[1]}, a^{[2]}\} \prec \{b^{[1]}, b^{[2]}\} \Longleftrightarrow \max_i a^{[i]} < \min_j b^{[j]}$.

We denote by $\mathcal{T}_B$ the family of *increasingly bilabelled unordered trees*, which contains all (non-empty) increasingly bilabelled unordered trees T of size $|T| \geq 1$ with label set $\mathcal{M} = \{0, 1, \ldots, 2|T| - 1\}$; we find it more convenient to use this label set instead of $\{1, 2, \ldots, 2|T|\}$ in order to state the bijections given later. In the first row of Fig. 1 all trees of $\mathcal{T}_B$ of size ≤ 3 are given.

Let us denote by $T_n^{[B]}$ the *number of different increasingly bilabelled unordered trees*, i.e., $T_n^{[B]} := |\mathcal{T}_B(n)|$. In the following we will prove that increasingly bilabelled unordered trees of size n are enumerated by the reduced tangent numbers $\tilde{E}_n$. We present a generating function proof, but more important, we also give a bijective proof by establishing a correspondence to a certain class of labelled strict binary trees introduced by Poupard [11]. We provide here (and also later in Section 3) also a generating function proof, since such an approach could be generalized easily to various other tree families (as, e.g., ordered trees or binary trees) and the differential equations obtained might be useful when determining the asymptotic behaviour of the corresponding numbers $T_n^{[B]}$ (and $T_n^{[D]}$ as defined in Section 3). However, it seems that apart from the here considered tree family no such explicit formulae for $T_n^{[B]}$ (and $T_n^{[D]}$) exist for other important tree families.

Strict binary trees (also called full or complete binary trees) are ordered trees, where each node has either two children (we may then speak about the left child and the right child) or no child (i.e., it is a leaf). An *increasingly labelled strict binary tree* T is a strict binary tree whose nodes are labelled by distinct integers (of a label set $\mathcal{M} = \mathcal{M}(T)$) in such a way that the label of a child node is always larger than the label of its parent node. We say that such a tree satisfies the *"right-smallest property"*, if for each non-leaf node v holds that the descendant of v (other than v) with the smallest label is contained in the right subtree of v.

We denote by $\mathcal{S}$ the tree family, which contains all *increasingly labelled strict binary trees* T of size $|T| \geq 1$ with *label set* $\mathcal{M} = \{1, 2, \ldots, |T|\}$ satisfying the *"right-smallest property"*. It has been shown in [11] that the number $|\mathcal{S}(2n-1)|$ of such trees of size $2n-1$ (it is immediate from the definition that the size of each such tree has to be odd) is given exactly by the reduced tangent number $\tilde{E}_n$. In the second row of Fig. 1 all trees of $\mathcal{S}$ of size ≤ 5 are given.

Theorem 2. *The number $T_n^{[B]}$ of increasingly bilabelled unordered trees of size n is given by the reduced tangent number: $T_n^{[B]} = \tilde{E}_n$.*

Generating function proof. Using standard combinatorial constructions the family $\mathcal{T}_B$ can be described by the following symbolic equation:

$$\mathcal{T}_B = \mathcal{Z}^\square * \left(\mathcal{Z}^\square * \textsc{Set}\big(\mathcal{T}_B\big)\right), \tag{3}$$

where $\mathcal{Z}$ denotes the *atomic class* (i.e., a single (uni)labelled node), $\mathcal{A} * \mathcal{B}$ denotes the *labelled product* and $\mathcal{A}^\square * \mathcal{B}$ the *boxed product* (i.e., the smallest label is constrained to lie in the $\mathcal{A}$ component) of the combinatorial classes $\mathcal{A}$ and $\mathcal{B}$, and $\textsc{Set}(\mathcal{A})$ denotes the class containing all *finite sets* of objects of $\mathcal{A}$; see [5].

We introduce the generating function

$$T_B(z) := \sum_{n\geq 1} T_n^{[B]} \frac{z^{2n}}{(2n)!}.$$

By an application of the *symbolic method* (see, e.g., [5]) Eq. (3) can be translated into the following differential equation for $T_B(z)$:

$$T_B''(z) = e^{T_B(z)}, \qquad T_B(0) = 0, \qquad T_B'(0) = 0. \tag{4}$$

It can be checked easily that the solution of (4) and its derivative are given as follows:

$$T_B(z) = \ln\left(1 + \tan^2\left(\frac{z}{\sqrt{2}}\right)\right), \qquad T_B'(z) = \sqrt{2}\tan\left(\frac{z}{\sqrt{2}}\right). \tag{5}$$

Thus Theorem 2 follows by extracting coefficients from (5):

$$T_n^{[B]} = (2n)![z^{2n}]T_B(z) = (2n-1)![z^{2n-1}]T_B'(z) = \tilde{E}_n, \quad n \geq 1. \quad \square$$

Bijective proof. We give a bijection between the objects of $\mathcal{T}_B(n)$ and the objects of $\mathcal{S}(2n-1)$. We do this by introducing a mapping φ,

$$\varphi : T \mapsto (S, l_0),$$

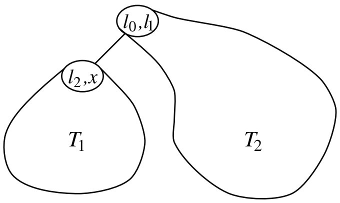
which maps any increasingly bilabelled unordered tree T of size n with label set $\mathcal{M} = \{l_0, l_1, \ldots, l_{2n-1}\}$ to the ordered pair (S, l_0), where S is an increasingly labelled strict binary tree with label set $\mathcal{M} \setminus \{l_0\} = \{l_1, l_2, \ldots, l_{2n-1}\}$ satisfying the "right-smallest property", and where l_0 is the smallest label of the label set $\mathcal{M}$.

The mapping φ will be defined recursively. In the following we always assume that $l_0 < l_1 < \cdots < l_{2n-1}$. Moreover, for each tree $T \in \mathcal{T}_B$, we may always assume that the subtrees of any node v are arranged from left-to-right such that the sequence containing the smaller labels of each child node of v is increasing.

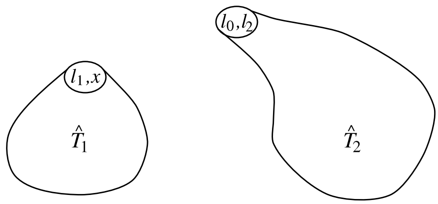
- $|T| = 1$: We simply define φ via:

$$\varphi : (l_0,l_1) \mapsto \big((l_1), l_0 \big).$$

- $|T| \geq 2$: Per definition the tree T has root node (l_0,l_1). Let us define by T_1 the subtree attached to the root of T containing (apart from the labels of the root of T) the smallest label l_2 (i.e., T_1 is the leftmost subtree of the root of T); furthermore let $T_2 := T \setminus T_1$. It holds that l_2 is the smaller label of the root of T_1.

$T =$ 

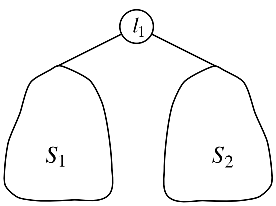
Now let us define by $\hat{T}_1$ and $\hat{T}_2$ the trees obtained from T_1 and T_2 by exchanging the labels l_1 and l_2.



It holds that $\hat{T}_1$ and $\hat{T}_2$ are both increasingly bilabelled unordered trees of sizes smaller than T. Furthermore the root of $\hat{T}_1$ is labelled by $\{l_1, x\}$, with $x \in \{l_3, \ldots, l_{2n-1}\}$, whereas the root of $\hat{T}_2$ is labelled by $\{l_0, l_2\}$. We can apply recursively the mapping φ to $\hat{T}_1$ and $\hat{T}_2$:

$$\hat{T}_1 \stackrel{\varphi}{\longmapsto} (S_1, l_1), \qquad \hat{T}_2 \stackrel{\varphi}{\longmapsto} (S_2, l_0),$$

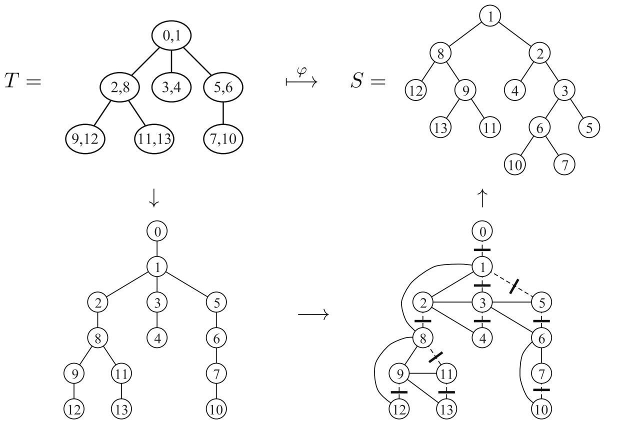
where S_1 and S_2 are increasingly labelled strict binary trees satisfying the "right-smallest property" with label sets $\mathcal{M}(\hat{T}_1) \setminus \{l_1\}$ and $\mathcal{M}(\hat{T}_2) \setminus \{l_0\}$, respectively.
We can construct now a labelled strict binary tree S by attaching the strict binary trees S_1 and S_2 as left and right subtree to a root node labelled l_1.

$S =$ 

Since S_1 and S_2 are increasingly labelled and l_1 is the smallest label of S it follows that S is an increasingly labelled strict binary tree with label set $\mathcal{M} \setminus \{l_0\}$. Since S_1 and S_2 satisfy the "right-smallest property" and the label l_2 is contained in S_2 (i.e., it is the root of S_2) the resulting tree S also satisfies the "right-smallest property". Thus we can define the mapping φ via:

$$\varphi : T \mapsto (S, l_0).$$

It is straightforward to give (again recursively) the inverse mapping φ^{-1} and thus to verify that φ is indeed a bijection; thus we omit here this step. Of course, Theorem 2 follows from this proof, where



The non-recursive construction:

(i) First replace every bilabelled node $\{a^{[1]}, a^{[2]}\}$ by two unilabelled nodes $a^{[1]}$ and $a^{[2]}$ such that $a^{[2]}$ becomes the child node of $a^{[1]}$.
(ii) Then cut-off root node; the child node of former root becomes the new root.
(iii) For each node v at an even level, which is not a leftmost child of its parent, we cut-off edge to its parent and add an edge to the left sibling.
(iv) For each node v at an odd level, which is not a leftmost child of its parent, we cut-off edge to its parent and add an edge to the left uncle (i.e., left sibling of the parent).
(v) For each non-root node v at an odd level, which is a leftmost child of its parent, we cut-off edge to its parent and add an edge to the grandparent, such that v becomes a left sibling of the former parent.

Fig. 2. An increasingly bilabelled unordered tree T of size 7 and the corresponding increasingly labelled strict binary tree of size 13 satisfying the "right-smallest property" obtained via the bijection φ as described in the bijective proof of Theorem 2.

one has the label set $\mathcal{M} = \{0, 1, \dots, 2n-1\}$ for any tree $T \in \mathcal{T}_B(n)$. This bijection is exemplified in Figs. 1 and 2. We remark that in Fig. 2 we illustrate the bijection φ by using a more direct non-recursive construction; however, we find it easier to verify the properties of φ (in particular that it is a bijection) when using the recursive description. $\square$

2.2. Hook-length formulae

Given a tree T of size n with distinguishable nodes (e.g., an ordered tree or an unordered labelled tree) and the label set $\mathcal{M} = \{0, 1, \dots, 2n-1\}$. When distributing the labels of $\mathcal{M}$ over the nodes of T such that each node gets exactly two labels, and if it further holds that each label of a child node is always larger than all labels of its parent node, we call this an *increasing bilabelling* of T. Let us denote by $\mathcal{L}^{[B]}(T)$ the *set of all increasing bilabellings* of T. When enumerating the number of increasing bilabellings of T the hook-lengths of the nodes of T appear naturally.

Lemma 3. *The number $|\mathcal{L}^{[B]}(T)|$ of different increasing bilabellings of a tree T of size n with distinguishable nodes is given as follows:*

$$|\mathcal{L}^{[B]}(T)| = \frac{(2n)!}{\prod_{v \in T} (2h_v(2h_v - 1))}.$$

Proof. The formula can be shown easily by using induction on the size $|T| = n$ of T.

- $n = 1$: There is one increasing bilabelling of T and this matches with the formula given.
- $n > 1$: We assume that the root of T has *out-degree* s (i.e., the root node has s children). Let us denote the subtrees of the root, which have corresponding sizes $k_1, \dots, k_s$, by $T_1, \dots, T_s$. It holds that

(after an order preserving relabelling) each of the subtrees $T_1, \ldots, T_s$ is itself an increasingly bilabelled tree. Taking into account that the root node of T is labelled by $\{0, 1\}$ and that the remaining nodes are distributed over the nodes of the subtrees one obtains:

$$|\mathcal{L}^{[B]}(T)| = \binom{2n-2}{2k_1, 2k_2, \ldots, 2k_s} \cdot |\mathcal{L}^{[B]}(T_1)| \cdot |\mathcal{L}^{[B]}(T_2)| \cdots |\mathcal{L}^{[B]}(T_s)|.$$

Using the induction hypothesis we further get:

$$|\mathcal{L}^{[B]}(T)| = \frac{(2n-2)!}{\prod_{j=1}^{s}(2k_j)!} \prod_{j=1}^{s} \frac{(2k_j)!}{\prod_{v \in T_j}(2h_v(2h_v-1))} = \frac{(2n)!}{\prod_{v \in T}(2h_v(2h_v-1))},$$

which completes the proof. $\square$

We introduce now the class $\hat{\mathcal{U}}$ containing all pairs $(T, L^{[B]}(T))$, with $T \in \mathcal{U}$ a labelled unordered tree and $L^{[B]}(T)$ an increasing bilabelling of T. We might consider the elements of $\hat{\mathcal{U}}$ as specifically trilabelled trees. Furthermore, we introduce the class $\hat{\mathcal{T}}_B$ containing all pairs $(T, L(T))$, with $T \in \mathcal{T}_B$ an increasingly bilabelled unordered tree and $L(T)$ a labelling of the nodes of T with distinct integers of $\{1, 2, \ldots, |T|\}$. Again we might consider the elements of $\hat{\mathcal{T}}_B$ as specifically trilabelled trees. However, it is immediate to see that the classes $\hat{\mathcal{U}}$ and $\hat{\mathcal{T}}_B$ are even isomorphic: they contain trilabelled unordered trees T, consisting of an ordinary labelling with $\{1, \ldots, |T|\}$ and of an increasing bilabelling with $\{0, 1, \ldots, 2|T| - 1\}$. In particular this implies that $|\hat{\mathcal{U}}(n)| = |\hat{\mathcal{T}}_B(n)|$, for arbitrary n. We have now all ingredients to finish the combinatorial proof of formula (1).

Proof of formula (1). It follows from the definition of $\hat{\mathcal{U}}$ and Lemma 3 that the size of $\hat{\mathcal{U}}(n)$ is given as follows:

$$|\hat{\mathcal{U}}(n)| = \sum_{T \in \mathcal{U}(n)} |\mathcal{L}^{[B]}(T)| = \sum_{T \in \mathcal{U}(n)} \frac{(2n)!}{\prod_{v \in T}(2h_v(2h_v-1))}.$$

On the other hand it follows from the definition of $\hat{\mathcal{T}}_B$ and Theorem 2 that the size of $\hat{\mathcal{T}}_B(n)$ is given as follows:

$$|\hat{\mathcal{T}}_B(n)| = n!|\mathcal{T}_B(n)| = n! T_n^{[B]} = n! \tilde{E}_n.$$

Since $|\hat{\mathcal{U}}(n)| = |\hat{\mathcal{T}}_B(n)|$ this shows the hook-length formula (1). $\square$

3. Double increasingly labelled trees

3.1. Enumeration results

We call a tree T a *double labelled* tree, if each node $v \in T$ has got an ordered pair $\ell_D(v) = (\ell^{[L]}(v), \ell^{[R]}(v))$ of integers (we may speak about the left label and the right label of v) such that the left labels as well as the right labels of two different nodes $v \neq w$ are different. We say that T is a double labelled tree with *left and right label set* $\mathcal{M}_L = \mathcal{M}_L(T) = \bigcup_{v \in T} \ell^{[L]}(v)$ and $\mathcal{M}_R = \mathcal{M}_R(T) = \bigcup_{v \in T} \ell^{[R]}(v)$, respectively; of course, $|\mathcal{M}_L| = |\mathcal{M}_R| = n$, for a tree T of size $|T| = n$. A double labelled tree T is called *increasing*, if it holds that the left label of a child node is always larger than the left label of its parent node as well as the right label of a child node is always larger than the right label of its parent node: $\ell_D(v) \prec \ell_D(w)$, whenever w is a child of v, where we use the relation $(a^{[L]}, a^{[R]}) \prec (b^{[L]}, b^{[R]}) \iff \left(a^{[L]} < b^{[L]} \text{ and } a^{[R]} < b^{[R]}\right)$.

We denote by $\mathcal{T}_D$ the family of *double increasingly labelled unordered trees*, which contains all (non-empty) double increasingly labelled unordered trees T of size $|T| \geq 1$ with left and right label sets $\mathcal{M}_L = \mathcal{M}_R = \{1, 2, \ldots, |T|\}$. In Fig. 3 all trees of $\mathcal{T}_D$ of size ≤ 3 are given.

Let us denote by $T_n^{[D]}$ the *number of different double increasingly labelled unordered trees*, i.e., $T_n^{[D]} := |\mathcal{T}_D(n)|$. In the following we will give a generating function proof as well as a bijective proof of the simple enumeration formula.

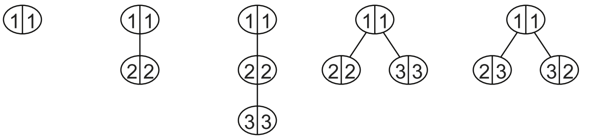


Fig. 3. Double increasingly labelled unordered trees of size up to three (in the figures the left and the right label of a node are always separated by a bar).

Theorem 4. *The number $T_n^{[D]}$ of double increasingly labelled unordered trees of size $n \geq 1$ is given as follows:*

$$T_n^{[D]} = \frac{(n-1)!n!}{2^{n-1}}.$$

Generating function proof. We use the decomposition of a tree $T \in \mathcal{T}_D$ of size $n \geq 2$ into the root node and its subtrees. Let us assume that the out-degree of the root of T is $s \geq 1$. After an order preserving relabelling the subtrees $T_1, \ldots, T_s$ are itself double increasingly labelled unordered trees of certain sizes $k_1, \ldots, k_s$. Since the root of T is always labelled by $(1, 1)$ and the remaining left labels and right labels are distributed over the nodes of $T_1, \ldots, T_s$ we obtain the following recurrence for the numbers $T_n^{[D]}$:

$$\begin{aligned} T_n^{[D]} &= \sum_{s\geq 1} \frac{1}{s!} \sum_{k_1+\cdots+k_s=n-1} \binom{n-1}{k_1, k_2, \ldots, k_s} \cdot \binom{n-1}{k_1, k_2, \ldots, k_s} \cdot T_{k_1}^{[D]} \cdot T_{k_2}^{[D]} \cdots T_{k_s}^{[D]}, \\ &\quad n \geq 2, \\ T_1^{[D]} &= 1. \end{aligned} \tag{6}$$

Note that the factor $\frac{1}{s!}$ is appearing in (6), since we are considering unordered trees.

Introducing the generating function

$$T_D(z) := \sum_{n\geq 1} T_n^{[D]} \frac{z^n}{(n!)^2},$$

the recurrence (6) leads to the following differential equation for $T_D(z)$:

$$zT_D''(z) + T_D'(z) = e^{T_D(z)}, \qquad T_D(0) = 0, \qquad T_D'(0) = 1. \tag{7}$$

It can be checked easily that the solution of this differential equation is given as follows:

$$T_D(z) = 2 \ln \left(\frac{1}{1 - \frac{z}{2}} \right). \tag{8}$$

Extracting coefficients from (8) shows thus the formula stated in Theorem 4:

$$T_n^{[D]} = (n!)^2 [z^n] T_D(z) = (n!)^2 \frac{2}{n2^n} = \frac{(n-1)!n!}{2^{n-1}}. \quad \square$$

Bijective proof. We will prove by combinatorial means that the numbers $T_n^{[D]}$ satisfy the following recurrence:

$$T_{n+1}^{[D]} = \binom{n+1}{2} T_n^{[D]}, \quad n \geq 1. \tag{9}$$

Of course, this will imply that $T_n^{[D]} = \prod_{k=2}^{n} \binom{k}{2} = \frac{n!(n-1)!}{2^{n-1}}$ as stated.

We do this by introducing a mapping ψ,

$$\psi : (T, \{v, w\}) \mapsto T',$$

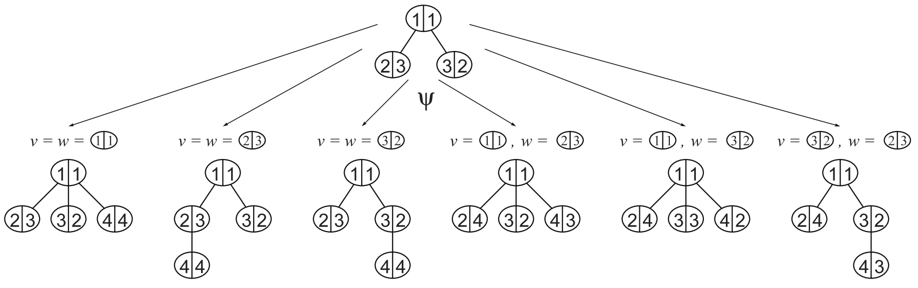


Fig. 4. A double increasingly labelled unordered tree T of size 3 and the resulting trees T' of size 4 obtained by applying the mapping ψ as described in the bijective proof of Theorem 4 to all 6 choices of multisets $\{v, w\}$ of nodes of T.

which maps any pair consisting of a double increasingly labelled unordered tree T of size n together with a multiset $\{v, w\}$ of nodes of T to a double increasingly labelled unordered tree T' of size $n+1$. In the following we describe the mapping ψ, where we distinguish, whether the multiset $\{v, w\}$ contains two different nodes or not.

- $v = w$: The tree T' is obtained from the tree T by attaching a new node labelled $(n+1, n+1)$ as a new child of v.
- $v \neq w$: We may assume that the right label of v is smaller than the right label of w: $\ell^{[R]}(v) < \ell^{[R]}(w)$. Let $q := \ell^{[R]}(w)$. We construct the tree T' by applying the following procedure to T.
  - (i) For each node $y \in T$ with a labelling $(\ell^{[L]}(y), \ell^{[R]}(y))$ such that $\ell^{[R]}(y) \geq q$ we replace this labelling by $(\ell^{[L]}(y), \ell^{[R]}(y) + 1)$ (i.e., we increase the right label by one).
  - (ii) Finally we attach a new node labelled $(n+1, q)$ as a new child of node v.

By construction, when starting with a tree $T \in \mathcal{T}_D(n)$ the mapping ψ yields a double increasingly labelled unordered tree T' of size $n+1$ and different choices of $(T, \{v, w\})$ lead to different trees T'.

It is straightforward to give the inverse mapping ψ^{-1} and thus to validate that the mapping ψ indeed yields a bijection between pairs $(T, \{v, w\})$ (with $T \in \mathcal{T}_D(n)$ and $\{v, w\}$ a multiset of nodes of T) and $T' \in \mathcal{T}_D(n+1)$; thus we omit to state ψ^{-1} explicitly. Since there are always $\binom{n+1}{2}$ choices for a multiset $\{v, w\}$ of nodes of a tree $T \in \mathcal{T}_D(n)$ this shows the recurrence (9). The mapping ψ is exemplified in Fig. 4. $\square$

We remark that the mapping ψ introduced above can be used to construct simple recursive algorithms to generate a random double increasingly labelled unordered tree of given size or to generate all double increasingly labelled unordered trees of given (small) size, respectively.

3.2. Hook-length formulae

In what follows it is advantageous to give an equivalent description of double labelled trees and variants. Namely, instead of thinking about trees, where each node gets two labels one might alternatively consider pairs of (uni)labelled trees of the same *shape* (we say that two trees are of the same shape if the corresponding underlying ordered trees are the same).[1]

To give an alternative description of the already studied tree family $\mathcal{T}_D$ and to show the hook-length formula (2) we introduce some notation. Let us denote by $\mathcal{L}$ and $\mathcal{I}$ the families of (arbitrary) *labelled ordered trees* and *increasingly labelled ordered trees* (always with label sets $\{1, 2, \ldots, |T|\}$ for any tree T), respectively. Moreover, we denote by $\hat{}$ the *"unordering mapping"*, which maps an ordered tree to the corresponding unordered tree (of course, one might alternatively think about building equivalence classes of ordered trees). The definition of the mapping $\hat{}$ can be extended to families of

[1] Actually this point of view yields the relations to problems studied in [3].

trees in an obvious way; e.g., $\hat{\mathcal{I}}$ gives then the family of increasingly labelled unordered trees, whereas $\hat{\mathcal{L}} = \mathcal{U}$, i.e., it gives the family of labelled unordered trees.

In the following we will always consider pairs (and triples) of labelled trees of the same shape. We use in this context the shortcut notation $(\mathcal{A} \mid \mathcal{B}) := \{(A, B) : A \in \mathcal{A}, B \in \mathcal{B}, \text{shape}(A) = \text{shape}(B)\}$ for tree families $\mathcal{A}$ and $\mathcal{B}$ (and the obvious extension to triples). When we apply the unordering mapping to the family $(\mathcal{I}_1 \mid \mathcal{I}_2) := \{(I_1, I_2) : I_1, I_2 \in \mathcal{I}, \text{shape}(I_1) = \text{shape}(I_2)\}$ then we get exactly the tree family $\mathcal{T}_D$ introduced in Section 3.1, i.e., $\mathcal{T}_D = \widehat{(\mathcal{I}_1 \mid \mathcal{I}_2)}$ (note that the mapping $\hat{}$ always has to act "simultaneously" to both components of equal shaped trees). It is now a simple, but important observation, that one gets isomorphic tree families when applying the unordering mapping $\hat{}$ to any of its components, i.e., that it holds

$$\widehat{(\mathcal{I}_1 \mid \mathcal{I}_2)} \cong (\hat{\mathcal{I}}_1 \mid \mathcal{I}_2) \cong (\mathcal{I}_1 \mid \hat{\mathcal{I}}_2).$$

Let us now consider the family $\widehat{(\mathcal{L} \mid \mathcal{I}_1 \mid \mathcal{I}_2)}$, which will play a central rôle in the proof of formula (2). Again it holds

$$\widehat{(\mathcal{L} \mid \mathcal{I}_1 \mid \mathcal{I}_2)} \cong (\hat{\mathcal{L}} \mid \mathcal{I}_1 \mid \mathcal{I}_2) \cong (\mathcal{L} \mid \hat{\mathcal{I}}_1 \mid \mathcal{I}_2) \cong (\mathcal{L} \mid \widehat{(\mathcal{I}_1 \mid \mathcal{I}_2)}). \tag{10}$$

Of particular interest are the second and the fourth family appearing in (10). Let us also recall the alternative description of these tree families as certain trilabelled trees. The family $(\hat{\mathcal{L}} \mid \mathcal{I}_1 \mid \mathcal{I}_2) = (\mathcal{U} \mid \mathcal{I}_1 \mid \mathcal{I}_2)$ can alternatively be considered as specifically trilabelled trees obtained by starting with labelled unordered trees, which are then equipped with a double increasing labelling. On the other hand the family $(\mathcal{L} \mid \widehat{(\mathcal{I}_1 \mid \mathcal{I}_2)}) = (\mathcal{L} \mid \mathcal{T}_D)$ can be considered as trilabelled trees obtained by starting with double increasingly labelled unordered trees, which are then additionally equipped with an arbitrary labelling. The combinatorial proof of formula (2) will be finished quickly.

Proof of formula (2). Due to (10) it follows that the number of trees of size n are the same for each of these families; in particular it holds

$$|(\mathcal{U} \mid \mathcal{I}_1 \mid \mathcal{I}_2)(n)| = |(\mathcal{L} \mid \mathcal{T}_D)(n)|. \tag{11}$$

First consider $(\mathcal{U} \mid \mathcal{I}_1 \mid \mathcal{I}_2)$. We get

$$\begin{aligned} |(\mathcal{U} \mid \mathcal{I}_1 \mid \mathcal{I}_2)(n)| &= \sum_{T \in \mathcal{U}(n)} |\{(I_1, I_2) : I_1, I_2 \in \mathcal{I}, \text{shape}(I_1) = \text{shape}(I_2) = \text{shape}(T)\}| \\ &= \sum_{T \in \mathcal{U}(n)} \left(\frac{n!}{\prod_{v \in T} h_v} \right)^2 = \sum_{T \in \mathcal{U}(n)} \frac{(n!)^2}{\prod_{v \in T} h_v^2}, \end{aligned}$$

where we used the well-known fact that the number of different increasing labellings of a tree T of size n with distinguishable nodes (and thus the number of increasingly labelled ordered trees of shape T) is given by $\frac{n!}{\prod_{v \in T} h_v}$ (see, e.g., [7,12]).

On the other hand by using Theorem 4 we obtain

$$|(\mathcal{L} \mid \mathcal{T}_D)(n)| = n!|\mathcal{T}_D(n)| = n! T_n^{[D]} = \frac{(n!)^2 (n-1)!}{2^{n-1}}.$$

Due to (11) the hook-length formula (1) follows from this. $\square$

4. From trees to forests

We consider now the families $\mathcal{U}$ and $\mathcal{U}_F$ of labelled unordered trees and forests, respectively, and show that each hook-length formula for trees naturally yields a corresponding hook-length formula for forests and vice versa. Let us assume that the following hook-length formula for trees in the general form as introduced by Han [8] and Chen et al. [1] holds (for $n \geq 1$):

$$\sum_{T \in \mathcal{U}(n)} \prod_{v \in T} \rho(h_v) = G(n), \tag{12}$$

with a certain function $G(n)$ and an arbitrary so-called *hook-weight function* $\rho : \mathbb{N}^+ \to \mathbb{R}$.

Consider a tree $T \in \mathcal{U}(n+1)$. When removing the root $r(T)$ of T then the resulting forest $F := T \setminus r(T)$ is, after an order preserving relabelling with $\{1, 2, \dots, n\}$, an element of $\mathcal{U}_F(n)$. Since the root of T can have any label of $\{1, 2, \dots, n+1\}$ it follows that each forest $F \in \mathcal{U}_F(n)$ is obtained exactly $n+1$ times when removing the root nodes of all trees $T \in \mathcal{U}(n+1)$ and applying a proper relabelling. Thus we get

$$\begin{aligned} G(n+1) &= \sum_{T \in \mathcal{U}(n+1)} \prod_{v \in T} \rho(h_v) = \rho(n+1) \sum_{T \in \mathcal{U}(n+1)} \prod_{v \in T \setminus r(T)} \rho(h_v) \\ &= (n+1)\rho(n+1) \sum_{F \in \mathcal{U}_F(n)} \prod_{v \in F} \rho(h_v), \end{aligned}$$

which implies the following hook-length formula for forests:

$$\sum_{F \in \mathcal{U}_F(n)} \prod_{v \in F} \rho(h_v) = \frac{G(n+1)}{(n+1)\rho(n+1)}. \tag{13}$$

In particular, the general relation between the hook-length formula (12) for trees and the corresponding formula (13) for forests yields, when using the hook-weight functions $\rho(n) = \frac{1}{2n(2n-1)}$ and $\rho(n) = \frac{1}{n^2}$, respectively, to the following hook-length formulae for forests $\mathcal{U}_F$ equivalent to (1) and (2) (valid for $n \geq 0$):

$$\sum_{F \in \mathcal{U}_F(n)} \prod_{v \in F} \left(\frac{1}{2h_v(2h_v - 1)} \right) = \frac{n!}{(2n)!} \tilde{E}_{n+1}, \tag{14}$$

$$\sum_{F \in \mathcal{U}_F(n)} \prod_{v \in F} \frac{1}{h_v^2} = \frac{(n+1)!}{2^n}. \tag{15}$$

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